Theory of compressional and transverse wave propagation in consolidated porous media

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A theory of compressional and shear wave propagation in consolidated porous media (rocks) is developed by extending ideas already introduced in connection with unconsolidated marine sediments. The consolidated material is treated as an elastic medium which exhibits a specific form of stress relaxation associated with grain boundaries and microcracks. The stress relaxation, which is linear in the sense that it obeys superposition, shows hysteresis, as characterized by a material response function. Two linear wave equations are derived, one for compressional and the second for shear waves, from which expressions for the wave speeds and attenuations are established. In both cases, the attenuation is found to scale with the first power of frequency, consistent with many observations of attenuation in sandstones, limestones, and shales; the wave speeds show weak logarithmic dispersion. These expressions for the wave speeds and attenuations satisfy the Kronig–Kramers dispersion relationships, as they must if the response of the medium to disturbances is to be causal. Some comments are offered on the nature of the material response, notably that it appears to be primarily associated with grain-boundary interactions occurring at a molecular level, rather than being related to the macroscopic properties of the material, such as density or porosity.
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LIST OF SYMBOLS

c_p	compressional wave speed (m/s)
c_s	shear wave speed (m/s)
c_{p0}	compressional wave speed in absence of intergranular stress relaxation (m/s)
c_{s0}	shear wave speed in absence of intergranular stress relaxation (m/s)
α_p	compressional attenuation coefficient (nepers/m)
α_s	shear attenuation coefficient (nepers/m)
β_p	compressional loss tangent
β_s	shear loss tangent
Q_p	quality factor, compressional waves
Q_s	quality factor, shear waves
χ_p	compressional dissipation coefficient
χ_s	shear dissipation coefficient
μ	dynamic rigidity modulus of skeletal frame
μ_p	compressional stress relaxation rigidity modulus
μ_s	shear stress relaxation rigidity modulus
ω	angular frequency
t	time
p	pressure fluctuation
$\underline{s}$	particle displacement
$\underline{v}$	particle velocity
ρ	density fluctuation
$\psi(t)$	velocity potential, compressional wave
$\psi_s(t)$	velocity potential, shear wave
$\underline{A}(t)$	vector potential
$\Psi(j\omega)$	Fourier transform of $\psi(t)$
$\Psi_s(j\omega)$	Fourier transform of $\psi_s(t)$
$h(t)$	compressional material response function
$h_s(t)$	shear material response function
$H(j\omega)$	Fourier transform of $h(t)$
$H_s(j\omega)$	Fourier transform of $h_s(t)$
n	compressional material response exponent $(0<n<1)$
m	shear material response exponent $(0<m<1)$
N	fractional porosity
ρ_0	bulk density of medium
ρ_f	density of pore fluid
$\underline{\rho_s}$	density of mineral solid
κ	bulk modulus of skeletal frame
κ	bulk modulus of medium
κ_f	bulk modulus of pore fluid
κ_s	bulk modulus of mineral solid

INTRODUCTION

A theoretical treatment of compressional and shear wave propagation in saturated, solid porous media such as rocks and consolidated marine sediments was developed in two classic papers by Biot.[1,2] Essentially, he considered that the pore fluid can move relative to the skeletal mineral frame, and found that two compressional waves are admissible, the so-called fast and slow waves, as well as a shear wave. The fast wave corresponds to the situation where the pore fluid and the solid material move essentially in phase, whereas they are in opposite phase in the slow wave. Although the slow wave is a propagating wave, it is very heavily attenuated relative to the fast wave.

Biot included dissipation in his theory, which he attrib-

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uted to viscosity of the pore fluid. As with any viscous medium, this gives rise to an attenuation coefficient that depends on frequency, f, scaling as f^2 at low frequencies and as $f^{1/2}$ above some threshold frequency. Correspondingly, his wave speeds are dispersive, increasing as $f^{1/2}$ at higher frequencies. Dissipation associated with the mineral frame, occurring for instance at grain boundaries, is not included in Biot's original model.

The question of the frequency dependence of the attenuation coefficient of waves in rocks is of considerable interest to the geophysics community. Many measurements of attenuation have been performed, both *in situ* and under laboratory conditions,[3] which provide strong evidence that the attenuation scales essentially as the first power of frequency, f^1, over the very broad frequency range from 10^{-2} to 10^7 Hz.[4] A discussion of these attenuation measurements, including a bibliography, is given by Johnston *et al.*[5] It has also been reported[6,7] that dispersion is negligible in rocks up to frequencies as high as 1 MHz, that is, the wave speed is essentially independent of frequency.

These experimental observations are not in accord with the predictions of the Biot theory.[1,2] The disparity stems from Biot's assumed loss mechanism, the viscosity of the pore fluid. Two alternative dissipation mechanisms are intrinsic loss in the minerals constituting the skeletal frame, although this is almost certainly negligible, and some form of dissipation arising from the relative movement between grain boundaries or crack surfaces. The latter type of dissipation cannot be identified as Coulomb friction, since this is a nonlinear mechanism for which there is no evidence[8] at the very low level of strain (10^{-6} or less) associated with wave propagation.

The purpose of this article is to develop a theory of seismic wave propagation in consolidated porous materials in which the dissipation, although arising from the relative movement of grain boundaries, is distinctly different from Coulomb friction. The loss mechanism in question was introduced recently by Buckingham[9–11] in connection with wave propagation in unconsolidated marine sediments. The new mechanism, which is strictly linear in that the principle of superposition is obeyed, takes account of stress relaxation occurring at grain-boundary contacts and leads to an attenuation that scales almost exactly as the first power of frequency and a phase speed that shows weak logarithmic dispersion. Both features are in accord with observations in saturated sediments and in rocks. Moreover, the predicted phase speed and attenuation satisfy the Kronig–Kramers[12,13] relationships, which must hold if the medium is to be causal, regardless of the physical mechanism responsible for the propagation.

An interesting outcome of the Biot theory[1,2] is the prediction of the slow compressional wave. Such a wave has been observed under controlled experimental conditions in consolidated porous media consisting of lightly fused glass beads saturated with water.[14,15] In unconsolidated media, it appears that the slow wave is either absent or negligible.[15] Since the slow wave is very difficult to detect, even in consolidated media, it is neglected entirely in the present analysis. In effect, this means that the pore fluid and the skeletal mineral frame are assumed to move together, which leads to a significant simplification in the theoretical development. This situation is, in fact, a special case of the Biot theory. Clearly, the slow wave could be included in the analysis by adopting the full formalism of the Biot theory, but with the new grain-boundary stress relaxation mechanism replacing the viscosity of the pore fluid. From a practical point of view, this more general approach would not lead to significantly different properties for the two primary waves, the fast wave and the shear wave, which is why it is not pursued here.

I. THE EQUATION OF MOTION

The analysis of wave propagation in a consolidated porous medium developed below follows lines similar to Buckingham's[9,10] treatment of waves in unconsolidated materials. Now, of course, the elasticity of the mineral frame must be included in the analysis, which is the main departure from the previous theoretical developments.

The porous medium is treated as a (macroscopically) homogeneous, isotropic elastic solid with shear modulus μ. Since the elasticity is associated exclusively with the mineral frame, μ is independent of the properties of the pore fluid. Thus, if the pore fluid were removed and replaced by a vacuum, μ could be measured directly using a standard laboratory technique. In fact, the shear modulus of most rocks is already available in the literature.[16] In setting up the wave equations, it is implicit that elements of volume are being considered in a way which is standard for a homogeneous medium. The dimensions of such volume elements are assumed to be small compared with the wavelengths of interest, but large in relation to the size of the pores. For most granular media of interest here, this treatment is valid for frequencies below several hundred kHz.

To begin, suppose that all dissipation is neglected. The composite material may then be treated as a simple elastic solid, the (linearized) equation of motion for which is[17]

$$\rho_0 \frac{\partial^2 \underline{s}}{\partial t^2} = \left(\kappa + \frac{4}{3}\mu \right) \text{grad div } \underline{s} - \mu \text{ curl curl } \underline{s}, \tag{1}$$

where ρ_0 is bulk density and κ is the bulk modulus of the medium, t is time, and the vector $\underline{s}$ is displacement. When the displacements are small, as is the case with wave propagation, the velocity, $\underline{v}$, can be expressed as the partial derivative of $\underline{s}$

$$\underline{v} = \frac{\partial \underline{s}}{\partial t}, \tag{2}$$

and the equation of motion can be expressed in terms of the velocity as

$$\rho_0 \frac{\partial^2 \underline{v}}{\partial t^2} = \left(\kappa + \frac{4}{3}\mu \right) \text{grad div } \underline{v} - \mu \text{ curl curl } \underline{v}. \tag{3}$$

Turning now to the question of dissipation, the stress relaxation terms are formulated exactly as in the treatment of compressional and shear waves in an unconsolidated medium.[10] Thus, the full equation of motion, including stress relaxation terms, is proposed as

$$\rho_0 \frac{\partial^2 \underline{v}}{\partial t^2} = \left(\kappa + \frac{4}{3}\mu \right) \text{grad div } \underline{v} - \mu \text{ curl curl } \underline{v}$$

$$+ \lambda_f \text{ grad div } \frac{\partial}{\partial t}[h(t) \otimes \underline{v}(t)]$$

$$+ \frac{4}{3}\eta_f \text{ grad div } \frac{\partial}{\partial t}[h_s(t) \otimes \underline{v}(t)]$$

$$- \eta_f \text{ curl curl } \frac{\partial}{\partial t}[h_s(t) \otimes \underline{v}(t)], \quad (4)$$

where $h(t)$ and $h_s(t)$ are material response functions for compressional and shear disturbances, respectively, and the symbol $\otimes$ denotes a temporal convolution. The stress relaxation terms in Eq. (4) will be familiar as representing viscous losses when the response functions are Dirac delta functions, since $\delta(t) \otimes \underline{v}(t) = \underline{v}(t)$. In this special case, η_f and λ_f would be the coefficients of shear and bulk viscosity, respectively. Now, viscosity is simply a form of internal friction that arises when layers of a fluid medium slide against one another. In such a medium the stress and strain rate follow each other instantaneously. By extension, the convolutions in Eq. (4) may be thought of as representing a generalized viscosity, or internal friction, that occurs in granular materials as particles move in relation to one another. In this situation, the stress–strain rate relationship depends on the history of the loading process, as represented by the response functions $h(t)$ and $h_s(t)$. With appropriate forms for the response functions, propagation behavior is obtained that differs significantly from that encountered in a viscous medium.

Although the stress relaxation terms in Eq. (4) are postulated as representing dissipation and dispersion associated with the relative motion of grain boundaries[18] in granular media, we still consider the material to be homogeneous and continuous as far as the wave analysis is concerned. This is consistent with the fact that the wavelengths of interest are very much greater than the size scale of the heterogeneities in the medium. As in the theoretical treatment of compressional and shear waves in unconsolidated materials,[10] the response functions are assigned the forms

$$h(t) = u(t) \frac{t_0^{n-1}}{t^n}, \quad 0 < n < 1, \quad (5)$$

and

$$h_s(t) = u(t) \frac{t_1^{m-1}}{t^m}, \quad 0 < m < 1, \quad (6)$$

where the unit step function, $u(t)$, ensures that the response of the medium is causal. The temporal coefficients t_0 and t_1 maintain the correct units of inverse time for the response functions, giving the convolutions the dimensions of velocity, as required.

The choice of an inverse-time power law for the response functions is not entirely arbitrary. Certain polymeric materials exhibit a form of stress relaxation that varies as t^{-n}, where n is positive, and in fact such behavior has been empirically represented by Nutting's equation.[19,20] As with granular materials, the polymers also show a constant-Q,[21] or equivalently, an attenuation that scales as the first power of frequency, suggesting that the granular media may show the same form of stress relaxation as the plastics, namely, t^{-n}.

II. WAVE EQUATIONS

The equation of motion [Eq. (4)] may be decomposed into two wave equations, one representing compressional and the other transverse disturbances, by expressing the velocity as the sum of the gradient of a scalar potential, ψ, and the curl of a zero-divergence vector potential $\underline{A}$. For shear waves of vertical polarization, $\underline{A}$ itself may be expressed in terms of a scalar shear potential ψ_s.

The argument is fairly standard and is the same as that followed by Buckingham,[10] except that now the elastic terms must be included in the equations. Rather than repeat the details of the derivation, the resultant wave equations are simply stated here:

$$\nabla^2 \psi - \frac{1}{c_{p0}^2} \frac{\partial^2 \psi}{\partial t^2} + \frac{\lambda_f}{\rho_0 c_{p0}^2} \frac{\partial}{\partial t} \nabla^2 [h(t) \otimes \psi(t)]$$

$$+ \frac{4\eta_f}{3\rho_0 c_{p0}^2} \frac{\partial}{\partial t} \nabla^2 [h_s(t) \otimes \psi(t)] = 0 \quad \text{(compressional)}, \quad (7)$$

and

$$\nabla^2 \psi_s - \frac{1}{c_{s0}^2} \frac{\partial^2 \psi_s}{\partial t^2} + \frac{\eta_f}{\rho_0 c_{s0}^2} \frac{\partial}{\partial t} \nabla^2 [h_s(t) \otimes \psi_s(t)]$$

$$= 0 \quad \text{(shear)}. \quad (8)$$

In these expressions, c_{p0} and c_{s0} are, respectively, the phase speeds that the compressional and shear waves would have if the stress relaxation were somehow turned off. Thus, c_{p0} and c_{s0} are given by the usual expressions for an elastic medium

$$c_{p0} = \sqrt{\frac{\kappa + \frac{4}{3}\mu}{\rho_0}}, \quad (9)$$

and

$$c_{s0} = \sqrt{\frac{\mu}{\rho_0}}. \quad (10)$$

As discussed below, the actual phase speeds differ from c_{p0} and c_{s0} because of the effective stiffness introduced into the medium by the grain-to-grain stress relaxation mechanism.

It is important to note that Eqs. (9) and (10) are expressed in the time domain, implying that the various elastic and dissipation coefficients appearing in these equations are real constants. That is to say, these coefficients are not permitted to be complex nor may they show any dependence on frequency.

To convert to the frequency domain, the wave equations must be Fourier transformed with respect to time. The resulting reduced wave equations are

$$\nabla^2 \Psi + \frac{\omega^2}{c_{p0}^2} \Psi + \frac{j\omega}{\rho_0 c_{p0}^2} [\lambda_f H(j\omega) + \tfrac{4}{3}\eta_f H_s(j\omega)] \nabla^2 \Psi = 0, \quad (11)$$

and

$$\nabla^2\Psi_s+\frac{\omega^2}{c_{s0}^2}\Psi_s+j\omega\frac{\eta_f}{\rho_0 c_{s0}^2}H_s(j\omega)\nabla^2\Psi_s=0, \tag{12}$$

where $j=\sqrt{-1}$, ω is angular frequency, and (Ψ,Ψ_s) are the Fourier transforms of the compressional and shear potentials (ψ,ψ_s). The transforms of the response functions (H,H_s) are the following standard forms:[22]

$$H(j\omega)=\frac{\Gamma(1-n)}{(j\omega t_0)^{1-n}}, \tag{13}$$

and

$$H_s(j\omega)=\frac{\Gamma(1-m)}{(j\omega t_1)^{1-m}}, \tag{14}$$

where $\Gamma(...)$ is the gamma function, which is essentially unity when n and m are very small numbers, as is the case for granular media.

By substituting Eqs. (13) and (14) into (11) and (12), the reduced wave equations may be expressed in the form

$$\nabla^2\Psi+\frac{\omega^2}{c_{p0}^2[1+(j\omega t_0)^n\chi_p+(j\omega t_1)^m(c_{s0}/c_{p0})^2\chi_s]}\Psi=0, \tag{15}$$

and

$$\nabla^2\Psi_s+\frac{\omega^2}{c_{s0}^2[1+(j\omega t_1)^m\chi_s]}\Psi_s=0, \tag{16}$$

where the dimensionless coefficients χ_p and χ_s are

$$\chi_p=\frac{\lambda_f\Gamma(1-n)}{\rho_0 c_{p0}^2 t_0}=\frac{4\mu_p}{3\rho_0 c_{p0}^2} \tag{17}$$

and

$$\chi_s=\frac{\eta_f\Gamma(1-m)}{\rho_0 c_{s0}^2 t_1}=\frac{\mu_s}{\rho_0 c_{s0}^2}. \tag{18}$$

Notice that these expressions define two new coefficients, μ_p and μ_s, which are stress relaxation rigidity moduli describing the additional stiffness of the medium that arises from the grain boundary interactions.

III. WAVE SPEEDS AND ATTENUATIONS

The phase speeds and attenuations of the compressional and shear waves are easily deduced from Eqs. (15) and (16), and in fact the former has already been discussed at some length in this context by Buckingham.[9] Since both equations have exactly the same form, only the latter is addressed here in the interest of brevity.

The solution of Eq. (16) for one-dimensional shear wave propagation in the x-direction is

$$\Psi_s=\Psi_{s0}\exp-j\frac{\omega}{c_{s0}\sqrt{1+(j\omega t_1)^m\chi_s}}|x|, \tag{19}$$

where Ψ_{s0} is the amplitude of the wave. It follows that the phase speed is

$$c_s=c_{s0}/\mathrm{Re}[1+(j\omega t_1)^m\chi_s]^{-1/2}, \tag{20}$$

and the attenuation coefficient is

$$\alpha_s=\frac{\omega}{c_{s0}}\mathrm{Im}[1+(j\omega t_1)^m\chi_s]^{-1/2}. \tag{21}$$

These expressions for the phase speed and attenuation are exact, since no approximations have yet been introduced (apart from linearizing the equation of motion). Notice that in the low frequency limit $(\omega\rightarrow 0)$ the phase speed reduces to c_{s0}, the speed in the absence of stress relaxation, and the attenuation is zero.

Since m is small compared with unity, the expressions in Eqs. (20) and (21) may be accurately approximated to first order in m. Based on an argument described by Buckingham,[9,10] the results are

$$c_s\approx c_{s0}\sqrt{1+\chi_s}\left[1+\frac{m\chi_s}{2(1+\chi_s)}\ln(|\omega|t_1)\right], \tag{22}$$

and

$$\alpha_s\approx\frac{m\pi}{4}\frac{\chi_s}{(1+\chi_s)^{3/2}}\frac{|\omega|}{c_{s0}}. \tag{23}$$

Thus, to this order of approximation, the attenuation coefficient scales as the first power of frequency and the phase speed shows very weak logarithmic dispersion, both in accord with observations of wave propagation in rocks.[7,23]

Now, the attenuation may alternatively be expressed in terms of the loss tangent, β_s, representing the imaginary part of the complex wave number relative to the real part. ($\beta_s=1/2Q_s$, where Q_s, is the quality factor, a parameter that is commonly used as a measure of attenuation in rock geophysics.) It is easily shown that

$$\beta_s=\frac{\alpha_s c_s}{\omega}\approx\frac{m\pi}{4}\frac{\chi_s}{1+\chi_s}, \tag{24}$$

from which it follows that the wave speed in Eq. (22) can be written as

$$c_s\approx c_{s0}\sqrt{1+\chi_s}\left[1+\frac{2\beta_s}{\pi}\ln(|\omega|t_1)\right]. \tag{25}$$

Similar results hold for the compressional wave:[9,10]

$$\beta_p\approx\frac{(\chi_p n\pi/4)+\chi_s(m\pi/3)(c_{s0}/c_{p0})^2}{1+\chi_p+\chi_s(4/3)(c_{s0}/c_{p0})^2} \tag{26}$$

and

$$c_p\approx c_{p0}\sqrt{1+\chi_p+\frac{4}{3}\left(\frac{c_{s0}}{c_{p0}}\right)^2\chi_s}\times\left[1+P\frac{2\beta_p}{\pi}\ln(|\omega|t_0)+S\frac{2\beta_p}{\pi}\ln(|\omega|t_1)\right], \tag{27}$$

$$P=\left[1+\frac{4m\chi_s}{3n\chi_p}\left(\frac{c_{s0}}{c_{p0}}\right)^2\right]^{-1}, \tag{27a}$$

$$S=\left[1+\frac{3n\chi_p}{4m\chi_s}\left(\frac{c_{p0}}{c_{s0}}\right)^2\right]^{-1}. \tag{27b}$$

These results for the phase speed and attenuation pairs are interesting because they satisfy the Kronig–Kramers relationships identically,[24] which is required if the response of the medium is to be causal. Although the question of causality is important in connection with pulse propagation, it has long been an unresolved issue in the context of materials which show an attenuation that scales as the first power of frequency.[25] Buckingham,[9] however, has shown that the impulse response of a dissipative medium of the type described above is strictly causal; that is to say, no anomalous behavior is observed in the received pulse around the origin in time.

If the weak frequency dispersion is neglected, the wave speeds in Eqs. (25) and (27) are approximated as

$$c_p \approx c_{p0}\sqrt{1+\chi_p+\frac{4}{3}\left(\frac{c_{s0}}{c_{p0}}\right)^2\chi_s} = \sqrt{\frac{\kappa+\frac{4}{3}[\mu+\mu_p+\mu_s]}{\rho_0}}, \tag{28}$$

and

$$c_s \approx c_{s0}\sqrt{1+\chi_s} = \sqrt{\frac{\mu+\mu_s}{\rho_0}}. \tag{29}$$

The loss tangents in Eqs. (24) and (26) can also be expressed in terms of the stress relaxation moduli μ_p and μ_s as follows:

$$\beta_p = \frac{\pi}{3}\frac{(n\mu_p+m\mu_s)}{[\kappa+\frac{4}{3}(\mu+\mu_p+\mu_s)]}, \tag{30}$$

and

$$\beta_s = \frac{m\pi}{4}\frac{\mu_s}{\mu+\mu_s}. \tag{31}$$

In the absence of a skeletal frame (i.e., $\mu=0$), Eqs. (28) to (31) reduce identically to the corresponding expressions for an unconsolidated sediment. According to Eqs. (28) and (29), the stress relaxation mechanism has the effect of raising the phase speeds above the values c_{p0} and c_{s0} that they would have had if the medium were lossless. This occurs because the stress relaxation terms lead, not only to attenuation, but also to a stiffening of the material, as is apparent from Eqs. (28) and (29), where the effective shear modulus is seen to be $(\mu+\mu_p+\mu_s)$ for compressional waves and $(\mu+\mu_s)$ for shear waves.

Mathematically, the additional stress relaxation stiffness, represented by the moduli μ_p and μ_s introduced in Eqs. (17) and (18), is characteristic of response functions that follow an inversetime power law. Such behavior would not occur if the stress relaxation were in the form of an exponential decay, nor, of course, would the attenuation scale as the first power of frequency. Note that, in the limiting case when n and m become vanishingly small, the attenuation [Eqs. (30) and (31)] goes to zero and the sole effect of the stress relaxation terms in the equation of motion is to raise the wave speeds by increasing the stiffness of the medium.

The approximations in Eqs. (25) and (24) have been compared[9] with the exact expressions in Eqs. (20) and (21), using values of the parameters that are representative of geophysical materials. It is difficult to distinguish the approximations from the exact forms over some 12 decades of frequency.

IV. EVALUATION OF THE GEOACOUSTIC PARAMETERS

Equations (28) to (31) for the compressional and shear wave speeds and attenuations contain two elastic moduli (κ, μ), four stress relaxation coefficients (μ_p, μ_s, n, m), and the density of the porous material, ρ_0. The values of the bulk modulus and the density may be derived from straightforward arguments, but the remaining five parameters are rather more difficult to evaluate, for reasons discussed below.

The density, ρ_0, is simply the weighted mean of the densities of the fluid and mineral components constituting the medium:

$$\rho_0 = N\rho_f + (1-N)\rho_s, \tag{32}$$

where ρ_f and ρ_s are the densities of the pore fluid and the mineral solid, respectively, and N is the porosity of the material. The bulk modulus, κ, of the porous medium depends on the bulk modulus of the mineral solid, κ_s, the bulk modulus of the pore fluid, κ_f, and the bulk modulus of the evacuated skeleton, $\bar{k}$. Based on the assumption that the skeletal frame and the pore fluid move together, Gassmann[26] derived the following expression[27] for the bulk modulus of the medium:

$$\kappa = \kappa_s\frac{\bar{\kappa}+q}{\kappa_s+q}, \quad q = \frac{\kappa_f(\kappa_s-\bar{\kappa})}{N(\kappa_s-\kappa_f)}. \tag{33}$$

The concept of a ''frame bulk modulus'' has been employed by Gassmann[26] and others to account for the ''softening'' effect on the overall bulk modulus of the contacts between grains. If the cementation forming the contact regions were as hard as the grains themselves, then the frame bulk modulus would equal the grain bulk modulus. If, on the other hand, the contact regions were extremely soft, then the frame bulk modulus would be close to zero. Generally, the effect of the contacts is quite different from either of these two extreme cases. In particular, the contacts appear to exhibit stress relaxation, that is, they are neither purely elastic nor purely dissipative, which of course is the basis of the wave theoretic analysis developed above.

The effect of the intergranular stress relaxation has already been taken into account in the wave theory through the terms involving λ_f and η_f in Eq. (4). These two coefficients are embedded, respectively, in the compressional and shear stress relaxation rigidity moduli, μ_p and μ_s, which appear in the expressions for the wave speeds and loss tangents. Since the compressional effect of the grain contacts is included in the wave theory, the frame bulk modulus, $\bar{k}$, is redundant and should be set to zero in Gassmann's expression for the bulk modulus of the medium. In this case, Gassmann's equation reduces to the familiar weighted mean for the bulk modulus of a two-phase medium:

$$\frac{1}{\kappa} = N\frac{1}{\kappa_f} + (1-N)\frac{1}{\kappa_s}. \tag{34}$$

Equation (34) in conjunction with the stress-relaxation wave theory imply that the composite medium is being treated as a fluid in which the grains are in suspension, with the compressional effects of the intergranular interactions being represented by the convolution involving the material response

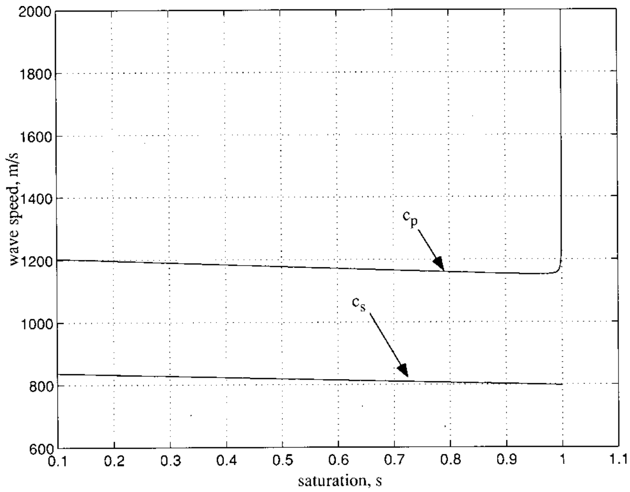

FIG. 1. Compressional and shear wave speeds computed from Eqs. (28), (29) and (32) to (35). The following parameter values, appropriate to the Murphy's Massillon sandstone, were used: $N=0.23$; $\rho_s=2660$ kg/m^3; $\rho_a=1$ kg/m^3; $\rho_w=998$ kg/m^3; $\kappa_s=3.6\times10^{10}$ Pa; $\kappa_a=1.16\times10^5$ Pa; $\kappa_w=2.19\times10^{10}$ Pa; $\bar{k}=0$.

function $h(t)$ in Eq. (4). Thus, this convolution term may, in a sense, be regarded as a substitute for the frame bulk modulus.

Incidentally, the elastic rigidity modulus, μ, representing the shear strength of the mineral frame cannot be set to zero because μ governs the shear-wave speed at very low frequencies [see Eq. (20)]. Since, in a consolidated material, the shear wave speed remains finite as the frequency is reduced, the rigidity modulus, μ, representing purely elastic interactions at the grain boundaries, must also be non-zero. To summarize, in a consolidated porous medium, the grain contacts give rise to stress relaxation, which introduces rigidity and dissipation, as well as effectively raising the bulk modulus of the two-phase medium, and they also provide elastic rigidity, which manifests itself as the stiffness of the mineral frame. In an unconsolidated medium such as sand or silt the only difference is that the elastic rigidity modulus is zero,[9,10] the implication being that in such materials there is no elastic frame. The lack of an elastic frame is consistent with the fact that an unconsolidated medium is incapable of supporting a tensile stress because the grains are, by definition, not bonded together. In the unconsolidated case, the rigidity that supports the propagation of shear waves is provided entirely by the stress relaxation occurring at the grain contacts.[10]

Returning to the consolidated case, in the event that some air is present in the pores, that is, the medium is not fully saturated, the density, ρ_f, and bulk modulus, κ_f, may be represented in terms of the fractional saturation, s, as follows:

$$\frac{1}{\kappa_f}=s\frac{1}{\kappa_w}+(1-s)\frac{1}{\kappa_a} \tag{35}$$

and

$$\rho_f=s\rho_w+(1-s)\rho_a, \tag{36}$$

where the subscripts ''a'' and ''w'' denote ''air'' and ''water,'' respectively. The effect of the level of saturation on the compressional and shear wave speeds is found by substituting the above expressions for the bulk modulus, κ, and the density, ρ_0, into Eqs. (28) and (29). Taking parameter values appropriate to Massilon sandstone, the material used by Murphy[28] in his experiments on wave speeds and attenuations, the curves shown in Fig. 1 are obtained. These curves are independent of n and m, but they do depend on μ, μ_p, and μ_s. These three parameters, which are independent of the saturation, have been chosen to fit the theoretical wave speeds to Murphy's measurements,[28] as presented in his Fig. 5. To obtain the fit, which is almost perfect, we have set $(\mu+\mu_s)=1.45\times10^{10}$ Pa and $\mu_p=8\times10^8$ Pa. Note that, from the wave speed data, it is not possible to assign values to μ and μ_s individually. In other words, the quantity $(\mu+\mu_s)$ plays the role of an effective elastic rigidity modulus as far as the wave speeds (and also the attenuations) are concerned.

It can be seen in Fig. 1 that both wave speeds show a gentle negative slope over most of the saturation range, which is due solely to the slight increase in the density, ρ_0, with decreasing gas content in the pores. As the saturation approaches 100%, nothing spectacular happens to the shear wave speed, but the compressional speed increases rapidly as a direct result of the dramatic increase in the bulk modulus of the pore fluid that occurs as the gas content is reduced to zero. These density and bulk modulus effects are well known and are consistent with Murphy's own interpretation[28] of his wave speed versus saturation data.

Murphy[28] also measured the attenuation (his Fig. 6) of the compressional and shear wave in Massilon sandstone. In principle, by fitting the theoretical expressions for the attenuation [Eqs. (29) and (30)] to these data, it should be possible to evaluate n and $m\mu_s$ (but not m and μ_s individually). Since all the rigidity moduli are independent of the level of saturation, it is evident from Murphy's Fig. 6 that n and m must both vary significantly with s. At present, as no physical model exists for the dependence of the material response indices on the saturation level, there is little to be gained by simply choosing empirical forms for n and m that match Murphy's attenuation data. In other types of rock, the attenuation behavior could be quite different.

A final comment on the two stress relaxation rigidity moduli is perhaps in order. It has been stated above that μ_p and μ_s, are independent of the level saturation, which is true provided sufficient moisture is present to lubricate the grain contacts. This will be the case when the saturation level is above approximately 2%. For lower levels of saturation, however, as found in moon rocks or vacuum-outgassed rocks, not even a monolayer layer of water is present on the surfaces of the pores and no lubrication exists at grain contacts. As a result, in an extremely dry, unlubricated material, μ_p and μ_s may be expected to take values that are significantly greater than those for the same material but with a higher moisture content. Moreover, since sliding at grain boundaries is strongly inhibited in such dry, unlubricated rocks, the material response indices, m and n, must decrease dramatically. In other words, the attenuation of compressional and shear waves in these extremely dry specimens is essentially zero.

V. CONCLUDING REMARKS

The internal stress relaxation mechanism proposed in this article has the potential for explaining much of the observed behavior of consolidated porous media (rocks). It differs from Coulomb friction[29] in that the new mechanism is linear, in the sense of obeying superposition, although the stress is not proportional to the rate of strain. Moreover, the theory leads naturally to an attenuation of compressional and shear waves that is proportional to the first power of frequency (i.e., a constant Q), as observed in wet and dry rocks. Since the stress relaxation is assumed to arise at grain boundaries and microcracks in the mineral frame, the theory provides a possible means of interpreting the effects of pore-fluid lubrication on wave speeds and attenuation. Similarly, the effects of both partial saturation and confining-pressure variations may also be investigated.

Numerous measurements of the geoacoustic properties of rocks, particularly sandstones, limestones, and oil-bearing shales, have been reported, many of which were taken under controlled laboratory conditions. Nevertheless, it would be very helpful to have available more comprehensive data sets containing direct measurements of wave speeds (compressional and shear) and attenuations, taken under varying dry and saturated conditions. As summarized by Johnston,[30] the attenuation is known to be less in air-dry rocks than in fully or partially saturated rocks; it is very much less in vacuum-outgassed rocks (e.g., lunar samples) than in air-dry rocks. It seems that the presence of one or two monolayers of water, corresponding to the air-dry condition, leads to a significant increase in the attenuation compared to an absence of water, obtained by vacuum outgassing.[31,32]

Interestingly, the introduction of benzene vapor instead of water to vacuum-outgassed sedimentary rocks produces little change in the attenuation.[32] Johnston[30] tentatively identifies the polar nature of water as being responsible for the observed rapid increase in attenuation when monolayers of water are introduced into the vacuum-outgassed rock. If this is so, it suggests that bonding forces on a molecular level give rise to the attenuation, which is consistent with the idea that the indices n and m, characterizing the material response functions in Eqs. (5) and (6), depend on microscopic interactions occurring at grain boundaries, rather than on macroscopic properties such as density or porosity. A similar suggestion has been made by Buckingham[9] in connection with attenuation in unconsolidated marine sediments. Based on these observations, it would appear that much information on the indices, n and m, and the stress relaxation moduli, μ_p and μ_c, could be obtained by further measurements of the attenuations and velocities in vacuum-outgassed rocks *vis-à-vis* air-dry rocks containing minute quantities of fluid.

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