

# Part A: Perturbation theory

W.R. Young <sup>1</sup>

March 2016

<sup>1</sup>Scripps Institution of Oceanography, University of California at San Diego, La Jolla, CA 92093-0230, USA. [wryoung@ucsd.edu](mailto:wryoung@ucsd.edu)

# Contents

<b>1 Algebraic perturbation theory</b>	<b>3</b>
1.1 An introductory example . . . . .	3
1.2 Iteration . . . . .	4
1.3 Singular perturbation of polynomial equations . . . . .	6
1.4 Double roots . . . . .	7
1.5 An example with logarithms . . . . .	8
1.6 Convergence . . . . .	10
1.7 Problems . . . . .	11
<b>2 Regular perturbation of ordinary differential equations</b>	<b>14</b>
2.1 The projectile problem . . . . .	14
2.2 A boundary value problem: belligerent drunks . . . . .	17
2.3 Failure of RPS — examples of singular perturbation problems . . . . .	19
2.4 Problems . . . . .	22
<b>3 Autonomous differential equations</b>	<b>24</b>
3.1 The phase line . . . . .	24
3.2 Population growth — the logistic equation . . . . .	25
3.3 The phase plane . . . . .	26
3.4 Matlab ODE tools . . . . .	30
3.5 The linear oscillator . . . . .	32
3.6 Nonlinear oscillators . . . . .	35
3.7 Problems . . . . .	37
<b>4 Regular perturbation of partial differential equations</b>	<b>45</b>
4.1 Potential flow round a slightly distorted cylinder . . . . .	45
4.2 Gravitational field of a slightly distorted sphere . . . . .	45
4.3 Problems . . . . .	45
<b>5 Boundary Layers</b>	<b>46</b>
5.1 Stommel's dirt pile . . . . .	46
5.2 Leading-order solution of the dirt-pile model . . . . .	48
5.3 Stommel's problem at infinite order . . . . .	50
5.4 A nonlinear Stommel problem . . . . .	53
5.5 Problems . . . . .	56
<b>6 More boundary layer theory</b>	<b>58</b>
6.1 Variable speed . . . . .	58
6.2 A second-order BVP with a boundary layer . . . . .	64

6.3	Other BL examples . . . . .	65
6.4	Problems . . . . .	68
<b>7</b>	<b>Multiple scale theory</b>	<b>70</b>
7.1	Introduction to two-timing . . . . .	70
7.2	The Duffing oscillator . . . . .	72
7.3	The quadratic oscillator . . . . .	74
7.4	Symmetry and the universality of the Landau equation . . . . .	76
7.5	The resonantly forced Duffing oscillator . . . . .	77
7.6	Problems . . . . .	81
<b>8</b>	<b>Rapid fluctuations</b>	<b>84</b>
8.1	A Lotka-Volterra Example . . . . .	84
8.2	Stokes drift . . . . .	85
8.3	Problems . . . . .	88
<b>9</b>	<b>Eigenvalue problems</b>	<b>90</b>
9.1	Regular Sturm-Liouville problems . . . . .	90
9.2	Properties of Sturm-Liouville eigenproblems . . . . .	93
9.3	Trouble with BVPs . . . . .	94
9.4	The eigenfunction expansion method . . . . .	95
9.5	Eigenvalue perturbations . . . . .	98
9.6	The vibrating string . . . . .	99
9.7	Problems . . . . .	101
<b>10</b>	<b>WKB</b>	<b>105</b>
10.1	The WKB approximation . . . . .	105
10.2	Applications of the WKB approximation . . . . .	106
10.3	An eigenproblem with a turning point . . . . .	110
10.4	Problems . . . . .	112
<b>11</b>	<b>Internal boundary layers</b>	<b>117</b>
11.1	A linear example . . . . .	117
<b>12</b>	<b>Initial layers</b>	<b>121</b>
12.1	The over-damped oscillator . . . . .	121
12.2	Problems . . . . .	122
<b>13</b>	<b>Boundary layers in fourth-order problems</b>	<b>124</b>
13.1	A fourth-order differential equation . . . . .	124
13.2	Problems . . . . .	126

## Part B: Asymptotic Expansions and Integrals

W.R. Young <sup>1</sup>

March 2016

<sup>1</sup>Scripps Institution of Oceanography, University of California at San Diego, La Jolla, CA 92093-0230, USA. [wryoung@ucsd.edu](mailto:wryoung@ucsd.edu)

# Contents

<b>1</b>	<b>Why integrals?</b>	<b>3</b>
1.1	The Airy function . . . . .	3
1.2	Recursion relations: the example $n!$ . . . . .	4
1.3	Special functions defined by integrals . . . . .	5
1.4	Elementary methods for evaluating integrals . . . . .	6
1.5	Complexification . . . . .	7
1.6	Problems . . . . .	10
<b>2</b>	<b>What is asymptotic?</b>	<b>14</b>
2.1	An example: the erf function . . . . .	14
2.2	The Landau symbols . . . . .	19
2.3	The definition of asymptoticity . . . . .	21
2.4	Stokes lines . . . . .	23
2.5	Problems . . . . .	23
<b>3</b>	<b>Integration by parts (IP)</b>	<b>26</b>
3.1	Dawson's integral and other examples . . . . .	26
3.2	The Taylor series, with remainder . . . . .	27
3.3	Large- $s$ behaviour of Laplace transforms . . . . .	29
3.4	Watson's Lemma . . . . .	31
3.5	Problems . . . . .	32
<b>4</b>	<b>Laplace's method</b>	<b>34</b>
4.1	An example — the Gaussian approximation . . . . .	35
4.2	Another Laplacian example . . . . .	37
4.3	Laplace's method with moving maximum . . . . .	39
4.4	Uniform approximations . . . . .	40
4.5	Problems . . . . .	42
<b>5</b>	<b>Fourier Integrals and Stationary phase</b>	<b>46</b>
5.1	Fourier Series . . . . .	46
5.2	Generalized Fourier Integrals . . . . .	51
5.3	The Airy function . . . . .	54
5.4	Problems . . . . .	55
<b>6</b>	<b>Dispersive wave equations</b>	<b>58</b>
6.1	Group velocity . . . . .	58
6.2	The 1D KG equation . . . . .	61
6.3	Problems . . . . .	66

<b>7</b>	<b>Constant-phase (a.k.a. steepest-descent) contours</b>	<b>67</b>
7.1	Asymptotic evaluation of an integral using a constant-phase contour . . . . .	68
7.2	Problems . . . . .	69
<b>8</b>	<b>The saddle-point method</b>	<b>70</b>
8.1	The Airy function as $x \rightarrow \infty$ . . . . .	70
8.2	The Laplace transform of a rapidly oscillatory function . . . . .	72
8.3	Inversion of a Laplace transform . . . . .	76
<b>9</b>	<b>Evaluating integrals by matching</b>	<b>77</b>
9.1	Singularity subtraction . . . . .	77
9.2	Local and global contributions . . . . .	78
9.3	An electrostatic problem — <b>H</b> section 3.5 . . . . .	81
9.4	Problems . . . . .	84